NASA TT F-9493

SCATTERING OF SOUND IN TURBULENT FLOW

A. M. Obukhov

NASA TT F-9493

Translation of "Über die Schallstreuung in der turbulenten Strömung". Doklady Akademii Nauk SSSR, Vol. XXX, No. 7, pp. 616-620, 1941

NATIONAL AERONAUTIC AND SPACE ADMINISTRATION
WASHINGTON D. C. JULY 1965

NASA TT F-9493

SCATTERING OF SOUND IN TURBULENT FLOW

A. M. Obukhov

(Presented by O. J. Schmidt, Member of the Academy, on December 7, 1940)

ABSTRACT

35781

An attempt is made to establish an equation for the scattering of sound within the turbulent atmosphere under the following assumptions: the distribution of the turbulent pulsations in space is random, statistically stationary and isotropic in the sense of Karmann. Expressions are derived for the amount of acoustic energy scattered into an elementary solid angle in a certain direction and per unit of time, as well as for the total scattered energy. Author

The investigations of the laws of sound-scattering under the conditions of a turbulent atmosphere is a current problem of meteorological acoustics. In the present report we have attempted to obtain an approximate solution of the problem using methods of /616*

* Note: Numbers in the margin indicate pagination in the original foreign text.

the statistical theory of turbulence, based on acoustical equations of media in motion.

§1. The basic equation for the propagation of sound in a medium in motion can be written in the following form:

$$\Delta\varphi - \frac{1}{c^2} D_t^2 \varphi = 0 \tag{1}$$

where ϕ is the potential of the acoustic wave* and

$$D_t = \frac{\partial}{\partial t} + (\bar{u}\,\overline{V});$$

D_t is the operator of complete differentiation with respect to the time; $\bar{u}$, the motion velocity vector of the fluid; c, the speed of sound. This equation was established by Andreyev (Ref. 1) by making a few simplifying assumptions.

Let us assume that the average flow velocity is equal to zero and that $\bar{u}$ is the instantaneous value of the pulsation velocity in the turbulent flow. Since the pulsation velocities under conditions within the earth's atmosphere are small with respect to the speed of sound, we will only retain terms in the equations, which are of an order corresponding to the smallest power of the small quantity $\mu = \frac{u}{c}$ and we will neglect terms of higher order.

By expanding the expression for the operator D_t^2 and by ignoring the terms that are small of the second order in equation (1), we

---

* In the present paper we will neglect the component of the sound field which does not have a potential which, in general, is present in the spreading of sound within a flow with vortices. The value of this component, in comparison to the potential component under conditions which correspond to the atmosphere, is not large.

obtain the "shortened" equation for the sound propagation:

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{2}{c^2}\left(\bar{u}, \text{degree}\,\frac{\partial\varphi}{\partial t}\right); \tag{2}$$

we will make use of this equation in the following.

§2. We will make the following assumptions concerning the field of turbulent pulsations: /617

1) The distribution of the pulsations in space is of a coincidental nature. The pulsations are statistically independent at points that are sufficiently distant from each other.

2) The distribution of the pulsations in space is statistically stationary; the reference moments among the pulsations at different points depend only on the vector connecting these points:

$$E\bar{u}(\bar{r}_1)\bar{u}(\bar{r}_2) = M(\bar{r}_1 - \bar{r}_2) \tag{3}$$

in which E is the symbol for mathematical expectation and M is the tensor of the reference moments.

3) The field of the turbulent pulsations is isotropic in the sense of Karmann (Ref. 2, 3).

The problem posed in the present paper consists of establishing a "local" equation for the scattering of sound, which determines the transformation of the sound wave through any "elementary volume" of the turbulent flow. Let us select an element having the form of a cube of volume V from the turbulent flow, whose dimensions must satisfy the following conditions: A) in comparison to the dimensions of the turbulent disturbances, the volume must be sufficiently large, so that the correlation at a distance of about one-fourth of the base length of the cube disappears for all practical purposes.

B) The dimensions of the elementary volume are such that the number of sound waves, that it can accomodate, is smaller than $\frac{1}{4\mu}$.

$\mu = \frac{u}{c}$ is the ratio of the average square of the pulsation velocity to the speed of sound.

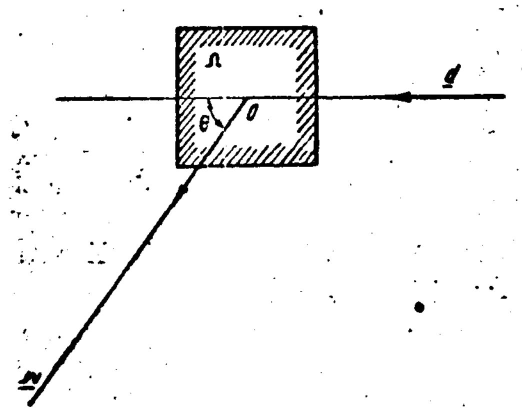

Figure 1.

Under ordinary conditions in the atmosphere $\mu \approx 0.003$ to 0.001, and $\frac{1}{\mu} \approx 300$ to $1{,}000$.

Since we are dealing with a *local problem*, we will assume, subject to some reservations, that the pulsation velocity outside of the volume considered is $\bar{u} \equiv 0$.

§3. Let the wave passing through be given by the equation

$$\varphi_0(\bar{r}, t) = A_0 e^{j(\omega t - (\bar{p}, \bar{r}))} \tag{4}$$

where $j = \sqrt{-1}$, $\frac{\omega}{p} = c$, $\bar{p}$ - is the wave vector and A_0 - is the amplitude.

We will look for the solution of equation (2), i.e., the function $\phi(\bar{r}, t)$, which approximates the function $\phi_0(\bar{r}, t)$ asymptotically as one departs from the "scattering" element V and which defines the undisturbed wave. Let us assume that $\phi = \phi_0 + \psi$. We then obtain the following differential equation for the "disturbance" ψ

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t} = \frac{2}{c^2}(\bar{u}, \mathrm{degree}\,\varphi_0) + \frac{2}{c^2}(\bar{u}, \mathbf{grad}\ \psi). \tag{5}$$

The disturbance disappears at infinity: $\psi(\bar{r}, t) \to 0$ for $r \to \infty$.

By integrating the inhomogeneous wave equation (5) by the method of retarded potentials, we obtain the following integral equation: /618

$$\psi(\bar{r}, t) = -\frac{1}{2\pi c^2}\iiint\limits_V \left(\frac{\bar{u}(\bar{\rho})}{R}, \mathrm{degree}\,\varphi_0\left(\bar{\rho}, t - \frac{R}{c}\right)\right) dV_{\bar{\rho}} - -\frac{1}{2\pi c^2}\iiint\limits_V \left(\frac{\bar{u}(\bar{\rho})}{R}, \mathrm{degree}\,\psi\left(\bar{\rho}, t - \frac{R}{c}\right)\right) dV_{\bar{\rho}} \tag{6}$$

in which $R = |\bar{r} - \bar{\rho}|$.

Let us assume that the center 0 of the volume V is the coordinate system origin; let us denote the unit vector which determines the direction of the scattered ray by $\bar{m}$, and let us call the distance between the center 0 and the point of observation R_0. For large values of R_0, the following development applies:

$$R = R_0 - (\overline{m}, \bar{\rho}); \ \frac{1}{R} = \frac{1}{R_0} + \frac{1}{R_0^2}(\overline{m}, \bar{\rho}).$$

By solving the integral equation obtained by the method of successive approximations and by again introducing the parameter μ, which is a small quantity, we obtain the desired solution in the form of a series:

$$\psi = \psi_1 + \psi_2 + \psi_3 + \ldots; \ \psi_2 = O(\mu^2); \ \psi_3 = O(\mu^3) \ldots$$

The convergence of this process is assured when the condition "B" is satisfied. Let us - considering the smallness of the quantity μ -

restrict ourselves to the first approximation. For large values of R_0 the following expression is obtained for it:

$$\begin{aligned}\psi_1(R_0, \bar{m}, t) &= \frac{A_0\omega}{2\pi c^2 R_0}\left(\bar{p}, \iiint_V \bar{u}(\bar{\rho})\, e^{j\omega(t-t_0)-j(\bar{p}-p\bar{m},\,\bar{\rho})}\, dV_{\bar{\rho}}\right) = \\ &= \frac{A_0\omega}{2\pi c^2 R_0} e^{j\omega(t-t_0)}\left(\bar{p}, \iiint_V \bar{u}(\bar{\rho})\, e^{-j(\bar{p}-p\bar{m},\,\bar{\rho})}\, dV_{\bar{\rho}}\right)\end{aligned} \tag{7}$$

in which $t_0 = \dfrac{R_0}{c}$.

For large distances from the center O, the effect of scattering of the acoustic wave by the element V is equivalent to the propagation of the spherical wave. Since the pulsations $\bar{u}(\bar{\rho})$ are of a coincidental nature, the wave described by the function ψ_1 is not coherent with the wave ϕ_0 passing through.

§4. In order to determine the sound energy, which is scattered by the element V in the direction of the vector $\bar{m}$, we determined the mean square of the amplitude ψ_1:

$$\begin{aligned} E|\psi_1|^2 &= E\psi_1\psi_1^* = \\ E\frac{A_0^2\omega^2}{4\pi^2c^4R_0^2}&\left((\bar{p}, \iiint_V\iiint_V \bar{u}(\bar{\rho}_1)\bar{u}(\bar{\rho}_2)e^{-j((\bar{p}-p\bar{m})(\bar{\rho}_1-\bar{\rho}_2))}\, dV_{\bar{\rho}_1} dV_{\bar{\rho}_2}\bar{p}\right) = \\ &= \frac{A_0\omega^2}{4\pi^2c^4R_0^2}\left(\bar{p}; \iiint_V\iiint_V M(\bar{\rho}_1-\bar{\rho}_2)\, e^{-j((\bar{p}-p\bar{m})(\bar{\rho}_1-\bar{\rho}_2))}\, dV_{\bar{\rho}_1} dV_{\bar{\rho}_2}\bar{p}\right)\end{aligned} \tag{8}$$

in which $M(\bar{\rho}_1 - \bar{\rho}_2) = E\bar{u}(\bar{\rho}_1)\cdot\bar{u}(\bar{\rho}_2)$ is the tensor of reference moments.

By applying condition "A", we can replace the first integration over the volume V by the integration over the entire space, without making a large error. Let us introduce the following additional notation

$$\Phi(\bar{k}) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty} e^{-j(\bar{k}\bar{r})}\, M(\bar{r})\, dV_{\bar{r}}\ ; \tag{9}$$

we then obtain the final expression for the mathematical expectation /619

$|\psi_1|^2$:

$$E|\psi_1|^2 = 2\pi \frac{A_0^2}{c^4} \cdot \frac{\omega^2}{R_0^2} (\bar{p}, \Phi(\bar{p} - p\bar{m})\,\bar{p})\, V_0 \tag{10}$$

in which V_0 is the volume of the scattering element V. The tensive function $\Phi(k)$ can be called the "spectral function" of the turbulent flow. It is closely related to the distribution of the turbulent pulsations among the frequencies. For the case of an incompressible, isotropic pulsation field, the spectral function is defined by only a scalar function which satisfies the following tensor equation:

$$\Phi(\bar{k}) = E\bar{u}^2 \left(F(k)\, I - F(k) \frac{\bar{k} \cdot \bar{k}}{k^2}\right) \tag{11}$$

in which I is the symbol for the unit tensor, $E\bar{u}^2$ is the mean square of the pulsation velocity.

Let us introduce the scattering angle θ, which is the angle between the direction of the wave passing through and the direction of the scattered ray $\bar{m}$. After some simple transformations, we obtain the following from equation (10) with the aid of equation (11)

$$E|\psi_1|^2 = \frac{A_0^2}{R_0^2} \cdot \frac{2\pi\omega^2}{c^4} \cdot \frac{\bar{u}^2}{c^2} \cdot F\left(4 \frac{\omega}{c} \sin^2 \frac{\theta}{2}\right) \cos^2 \frac{\theta}{2} \cdot \tag{12}$$

It is also advantageous to introduce a dimensionless spectral function f when using the concept of a linear scale for turbulence. For this purpose, let us set

$$F(k) = l^3 f(lk)$$

in which l is some characteristic length which is connected with the nature of the turbulent flow. Equation (12) can be rewritten in the following way:

$$E|\psi_1|^2 = \frac{A_0^2}{R_0^2}(2\pi)^4\left(\frac{l}{\lambda}\right)^4\mu^2 f\left(8\pi\frac{l}{\lambda}\sin^2\frac{\theta}{2}\right)\cos^2\frac{\theta}{2}\cdot\frac{V}{l} \qquad (13)$$
$$\mu^2 = \frac{E(\bar{u}^2)}{c^2}.$$

The sound energy is proportional to the second power of the amplitude. Therefore, an amount of energy which is proportional to $E|\psi_1|^2 R_0^2 d\Omega$ is scattered into an element having the solid angle $d\Omega$.

Let us set $V_0 = S\Delta x$, in which Δx is an element of the path traversed by the incoming wave. Let us call the energy flux U. Thus, $U_0 = \chi A_0^2 S$ is the energy flux passing through and $U' = \chi\iint_\Omega |\psi^2| R_0^2 d\Omega$ is the flux of the scattered energy. χ is a proportionality constant. By multiplying both parts of equation (13) by $\chi R_0^2 \, d\Omega_\theta$, we obtain the sound energy scattered into the element having a solid angle $d\Omega$ in the direction θ per unit of time:

$$dU' = U_0 (2\pi)^4 \mu^2\left(\frac{l}{\lambda}\right)^4 f\left(8\pi\frac{l}{\lambda}\sin^2\frac{\theta}{2}\right)\cos^2\frac{\theta}{2}\frac{\Delta x}{l}\, d\Omega_\theta. \qquad (14)$$

By integrating over the sphere, we obtain the total amount of scattered energy

$$U' = U_0 \mu^2 f_1\left(\frac{l}{\lambda}\right)^4\frac{\Delta x}{l},$$

where

$$f_1\left(\frac{l}{\lambda}\right) = (2\pi)^4\iint_\Omega f\left(8\pi\frac{l}{\lambda}\sin^2\frac{\theta}{2}\right)\cos^2\frac{\theta}{2}\, d\Omega_\theta \qquad (15)$$

Equations (14) and (15) are the complete solution of the problem. /620

It is natural to establish the hypothesis, that the scattered energy U′ comes from the fundamental flow having the energy U_0, which is the reason the fundamental flow U_0 experiences an attenuation when it penetrates the thickness of the turbulent layer in the atmosphere.

The attenuation coefficient of the energy of the fundamental flow due to scattering can be expressed in the following way:

$$q=\mu^2\left(\frac{l}{\lambda}\right)^4 f_1\left(\frac{l}{\lambda}\right)\frac{1}{l}.$$

In order to obtain numerical results, it is necessary to know the dimensionless spectral function f, the value of the turbulent scale l and the mean square of the pulsation. These quantities can be obtained by means of appropriate experimental investigations under natural atmosphere conditions.

Institute for Theoretical Geophysics.
Academy of Sciences of the USSR, Moscow

Received
December 14, 1940

REFERENCES

1. Andreyev, H. H., and Rusakov, A. G. Acoustics of a Moving Medium (Akustika dbizhyshcheysya sredy). 1934.

2. Karmann Usp. Fiz. Nauk, XXI, No. 1, 1939.

3. Loytsyanskiy, L. G. Tr. Ts AGI (Trudy Tsentral'nyy Aerogidrodinamicheskiy Institut), No. 440, 1939.

Translated by
K. Lehmann

Scientific Translation Service
1144 Descanso Drive
La Canada, California